J. Fluid Mech. (1967), *vol.* 30, *part* 1, *pp.* 127–147
Printed in Great Britain

The stability of rotating flows with a cylindrical free surface

By T. J. PEDLEY
Department of Mechanics, The Johns Hopkins University

(Received 18 October 1966 and in revised form 7 February 1967)

The stability to small inviscid disturbances of a rotating flow, whose velocity components in cylindrical polars (r, θ, z) are $(0, V(r), 0)$, is investigated when one boundary of the flow $(r = b)$ is a free surface under the action of surface tension (γ), and the other is either at infinity, or a rigid cylinder $(r = a \neq b)$, or at the axis $(r = 0)$. The free surface may be the inner or the outer boundary. A necessary and sufficient condition for stability to axisymmetric disturbances is derived, which requires that Rayleigh's criterion of increasing circulation be satisfied, and otherwise depends only on b, $V(b)$, γ and the density of the swirling liquid. This condition may be extended to include non-axisymmetric disturbances when $V \propto 1/r$ and when $V \propto r$ although in the latter case it is no longer a necessary one. It is shown that, in the case $V \propto r$, as well as $V \propto 1/r$, the 'most unstable' disturbance on a rotating column of fluid will be non-axisymmetric if the rotation speed at the surface is sufficiently great. Several applications of the theory are suggested, and a possible experiment to test it is described.

1. Introduction

The purpose of this paper is to investigate the hydrodynamic stability of a general swirling flow, velocity components $(0, V(r), 0)$ in cylindrical polar co-ordinates (r, θ, z), when one boundary of the flow, the cylinder $r = b$, is a free surface under the action of surface tension. This surface may be the outer or the inner boundary of the flow, and the other boundary is in general taken to be a concentric rigid cylinder of radius a $(a \neq b)$, where the two special cases $a = \infty$—i.e. the situation is that of an unbounded fluid (e.g. water) swirling round a hollow (air) core—and $a = 0$—there is a swirling column of (say) water in air, unattached to any rigid boundary—are included, and must sometimes be discussed separately. The analysis will be restricted to small disturbances (so that the equations of motion may be linearized) and viscosity will be ignored, as will the inertia of the 'air'.

The problem is seen to be an amalgamation of two classical stability problems. (i) The first is the stability of inviscid rotating flows between two concentric, rigid, circular cylinders, where a necessary and sufficient condition for stability to axisymmetric disturbances is that the square of the circulation should nowhere decrease outwards—i.e.

$$\Phi(r) = \frac{1}{r^3}\frac{d}{dr}(r^2V^2) \geqslant 0 \tag{1.1}$$

everywhere (this result was first given by Rayleigh (1916); see Lin (1955, p. 50) for a rigorous proof). The result is *not* valid for non-axisymmetric disturbances. (ii) The other relevant classical problem is the stability of a cylindrical fluid surface (radius b) under the action of surface tension, when there is no rotation—the only motion in the undisturbed state is a uniform axial velocity which may be removed from the problem by a suitable choice of co-ordinates. Here a necessary and sufficient condition for stability to a *general* disturbance, with wave-numbers k in the axial direction and n in the azimuthal direction—the z- and θ-dependence of all perturbation flow quantities may be written $\exp[i(kz+n\theta)]$—is that

$$1-n^2-k^2b^2 \leqslant 0. \tag{1.2}$$

This result is also due to Rayleigh (1879). Note that, since n is an integer (for quantities to be periodic in θ), only axisymmetric disturbances ($n = 0$) *can* be unstable, and those for which $k < 1/b$ *will* be unstable.

The combined problem has been attempted by several authors, but only when the basic rotation takes one of two simple forms, either $V(r) = \Gamma/r$ (potential vortex flow) or $V(r) = \Omega r$ (solid-body rotation). The present paper seeks to extend their results to general forms of $V(r)$, for both axisymmetric and non-axisymmetric disturbances, although success in the latter case has been limited. The results of previous authors, and the new ones derived in this paper, are best presented in the form of table 1, where the conditions for stability to all types of disturbance—axisymmetric ($n = 0$), plane ($k = 0$), and general ($n \neq 0$, $k \neq 0$)—of different basic swirl velocity fields $V(r)$ are displayed, with an indication of whether the given condition is necessary for stability, or sufficient, or both. The sources of these results are also given (the quoted paper by Hocking (1960) is a verification and extension of earlier work by Hocking & Michael (1959)), and an asterisk means that the result is believed to be new. Much of the notation has already been introduced; $\Phi(r)$ is defined by (1.1), and the quantities A_+ and A_- are defined as follows:

$$A_{\pm} = \frac{\gamma}{\rho b^2}\left\{1-n^2-k^2b^2 \pm \frac{\rho b V^2(b)}{\gamma}\right\}, \tag{1.3}$$

where ρ is the density of the fluid, and γ is the surface tension of the free surface. A_+ is the relevant quantity when the free surface is the outer boundary of the flow ($b > a$), and A_- is the relevant quantity when the free surface is the inner boundary ($b < a$). Whenever results for the two cases are combined in one expression (e.g. $A_{\pm}$), the upper sign will refer to a free outer boundary, and the lower sign to a free inner boundary.

We may notice that in all situations where $A_{\pm} \leqslant 0$ is a necessary and sufficient condition for stability, then, when the outer boundary is free, the cylindrical free surface is destabilized by rotation; and, when the inner boundary is free, it is stabilized. For instance, an axisymmetric disturbance with wave-number k greater than $1/b$ is stable in the absence of rotation (from (1.2)), but (1.3) shows that A_+ can be positive for *any* k, when the outer boundary is free, provided that the basic swirl velocity at $r = b$ is large enough. In addition, some non-axisymmetric disturbances, stable without rotation, become unstable when rotation is included, and indeed, for sufficiently large $V(b)$, the most unstable disturbance (that mode in which instability is in fact exhibited) may be non-axisymmetric; Ponstein (1959) shows this to be the case when $V = \Gamma/r$, and it is shown in § 6 below that it is also true when $V = \Omega r$. Similarly, we see that, when the inner boundary is free and

$$\rho b V^2(b)/\gamma \geqslant 1,$$

A_- can never be negative, and the surface must be stable to all disturbances.

Basic velocity field $V(r)$	Type of disturbance: Axisymmetric ($n = 0$)	Plane ($k = 0$)	General ($n \neq 0$, $k \neq 0$)
Potential vortex flow Γ/r	$A_\pm < 0$ necessary and sufficient	$A_\pm < 0$ necessary and sufficient	$A_\pm < 0$ necessary and sufficient
		Ponstein (1959)	
Solid-body rotation Ωr	$A_\pm \leqslant 0$ necessary and sufficient Hocking (1960) (case (a)) Rosenthal (1962) (case (b))	$\dfrac{\rho b V^2(b)}{\gamma} < n(n+1)$ necessary and sufficient Hocking (1960) (case (a) only). Case (b) is stable*	$A_\pm \leqslant 0$ sufficient*
General: $\Phi(r) \geqslant 0$ everywhere	$A_\pm \leqslant 0$ necessary and sufficient*	$A_\pm \leqslant 0$ *and* $0 \leqslant \dfrac{d}{dr}\left(\dfrac{V}{r}\right) \leqslant \dfrac{8k^2r^3V(r)}{n^2}$ sufficient (not very helpful)*	
General: $\Phi(r) < 0$ somewhere	Unstable*		

TABLE 1. Conditions for stability of the rotating flow given in the left-hand column, to disturbances given in the top row, with sources. Cases (a) and (b) refer to the free surface being the outer and inner boundary respectively. Asterisk indicates new result.

$$\Phi(r) \equiv \frac{1}{r^3}\frac{d}{dr}(r^2V^2); \quad A_\pm = \frac{\gamma}{\rho b^2}\left(1 - n^2 - k^2b^2 \pm \frac{\rho b V^2(b)}{\gamma}\right)$$

The exact analysis involved in deriving the asterisked results of table 1 is somewhat lengthy, so, before it is undertaken, a heuristic argument will be presented (in § 3) showing the plausibility of the results for axisymmetric disturbances to general swirling flows. The complete equations and boundary conditions are set down in § 2. In § 4 a sufficient condition for stability is derived in as general a manner as possible, although it can be usefully extended to non-axisymmetric disturbances only in the case of solid-body rotation. Then in § 5 the necessary condition for stability to axisymmetric disturbances, of flows with a general $V(r)$, is rigorously derived. Section 6 consists merely of a demonstration that the most unstable disturbance for solid-body rotation, when the outer boundary is free, is in some circumstances non-axisymmetric, while § 7 considers possible applications of the results, suggests an experiment to test them, and assesses the hitherto neglected effect of viscosity on the stability criteria.

2. Equations and boundary conditions

Equations

Let us assume a small disturbance to the basic flow, writing the velocity components as $(\tilde{u}, V(r)+\tilde{v}, \tilde{w})$ and the pressure as $\rho[P(r)+\tilde{p}]$, where ρ is the density of the water, and $P(r)$ is given by

$$\frac{dP}{dr} = \frac{V^2}{r} \tag{2.1}$$

(the basic flow equation). If we neglect squares and products of the disturbance quantities, the full inviscid equations of motion may be linearized. The linear equations are separable, and the general solution for $\tilde{u}$ (for example) is expressible as a sum of components like

$$\tilde{u} = u(r)\exp[i(\alpha t + kz + n\theta)];$$

because of the linearization, it is possible to consider just one such Fourier component at a time, superposing the solutions later if it is required to fit them to given initial conditions. Thus we may assume that the quantities $\tilde{u}$, $\tilde{v}$, $\tilde{w}$, $\tilde{p}$ take the form:

$$(\tilde{u}, \tilde{v}, \tilde{w}, \tilde{p}) = [u(r), v(r), w(r), p(r)]\exp[i(\alpha t + kz + n\theta)],$$

where real parts are assumed to be taken when results are to be applied to the real quantities $(\tilde{u}, \tilde{v}, \tilde{w}, \tilde{p})$, and where for a real disturbance k is positive and n is an integer.

If we now follow Howard & Gupta (1962) and eliminate $v(r)$, $w(r)$ and $p(r)$ from the linearized equations, we are left with the following single ordinary differential equation for $u(r)$:

$$\sigma^2 \frac{d}{dr}\left[\frac{S}{r}\frac{d}{dr}(ru)\right] - u\left\{\sigma^2 + \sigma r \frac{d}{dr}\left[S\left(\frac{1}{r}\frac{d\sigma}{dr} + \frac{2nV}{r^3}\right)\right] - k^2 S\Phi\right\} = 0, \tag{2.2}$$

in which

$$\sigma = \alpha + \frac{nV}{r}, \tag{2.3}$$

$$S = \frac{r^2}{n^2 + k^2 r^2}, \tag{2.4}$$

and $\Phi(r)$ is given by (1.1). An equation for $p(r)$ in terms of $u(r)$ (necessary for applying the boundary conditions) is

$$\frac{irp}{\sigma^2 S} = \frac{d}{dr}\left(\frac{ru}{\sigma}\right) - \frac{2nV}{r\sigma^2}u. \tag{2.5}$$

Boundary conditions

The equation (2.2) is of second order, so that one condition at each end of the range of r, (b, a), will be sufficient to determine the solution. First of all the condition at $r = a$: if $a = 0$ or a finite constant, it will be seen that no radial velocity is possible there, so the condition to be applied is $u = 0$. In the case where a is infinite, we must at least apply the condition that the energy density of the disturbance tends to zero. The first boundary condition is thus

$$\left.\begin{aligned} u &= 0, \quad \begin{cases} r = 0, \\ r = a \text{ (finite)}, \end{cases} \\ u &= o(r^{-\frac{1}{2}}), \qquad r \to \infty. \end{aligned}\right\} \tag{2.6}$$

The second boundary condition must be applied at the free surface, which we may take to be given by

$$r = b + \tilde{\delta},$$

where

$$\tilde{\delta} = \delta \exp\left[i(\alpha t + kz + n\theta)\right] \tag{2.7}$$

and

$$|\delta| \ll b$$

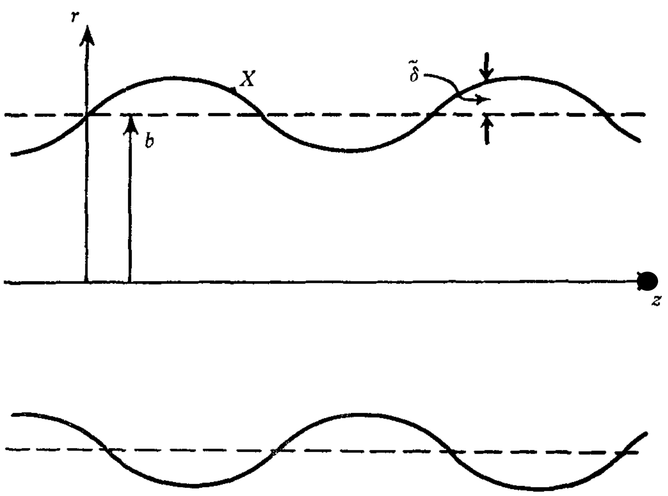

FIGURE 1. Axial cross-section of the disturbed surface.

(see figure 1). The dynamic boundary condition states that the total pressure in the fluid at $r = b + \tilde{\delta}$ is equal to the apparent pressure there due to the effect of surface tension. In other words

$$\rho(P + \tilde{p}) = \pm\gamma\left(\frac{1}{R_1} + \frac{1}{R_2}\right) \quad \text{on} \quad r = b + \tilde{\delta}, \tag{2.8}$$

where γ is the surface tension, and R_1 and R_2 are the principal radii of curvature of the surface. The plus sign is to be taken when the outer boundary is free and the minus sign when the inner boundary is free. To the first order in $|\delta/b|$, the right-hand side of (2.8) is then

$$\pm\gamma\left\{\frac{1}{b} - \frac{\tilde{\delta}}{b^2}(1 - n^2 - k^2b^2)\right\},$$

when $\tilde{\delta}$ is given by (2.7). The value of P on the surface in the undisturbed state must be $\pm\gamma/\rho b$ in order that equilibrium may be possible. Hence, at a general radius r, (2.1) tells us that

$$P = \pm\frac{\gamma}{\rho b} + \int_b^r \frac{V^2(r)}{r}\,dr.$$

On the disturbed surface, therefore,

$$P = \pm\frac{\gamma}{\rho b} + \frac{V^2(b)}{b}\tilde{\delta}. \tag{2.9}$$

Thus (2.8) becomes

$$\begin{aligned} p &= \mp\frac{\gamma}{\rho b^2}\left\{1-n^2-k^2b^2 \pm \frac{\rho b V^2(b)}{\gamma}\right\}\delta \\ &= \mp A_\pm \delta \end{aligned} \tag{2.10}$$

from the definition (1.3) of $A_\pm$.

But there is also a kinematic boundary condition to be satisfied: the normal velocity on the free surface when determined from the rate of change of $\tilde{\delta}$ must be the same as when it is determined from the velocity components $(\tilde{u}, \tilde{v}, \tilde{w})$. To the first order in $|\delta/b|$ this requires that

$$\tilde{u}(r=b) = \frac{D\tilde{\delta}}{Dt} = i\sigma\tilde{\delta},$$

so (2.10) becomes

$$i\sigma p = \mp A_\pm u \quad \text{on} \quad r = b. \tag{2.11}$$

(2.6) and (2.11) are the boundary conditions to be applied to any solution u of (2.2); p is given in terms of u by (2.5).

The problem is to find the eigenvalues for α^2 ($\alpha^2 > 0$ means an oscillating stability; $\alpha^2 < 0$ or complex means instability or overstability) for every integer value of n and for every positive value of k with all forms of the function $\Phi(r)$ and either sign for $A_\pm$.

3. A heuristic argument

Rayleigh's original argument that the square of the circulation must increase outwards for a swirling flow to be stable to axisymmetric disturbances was based on a consideration of the change in centrifugal potential energy when a ring of fluid at radius r_1 is interchanged with one at r_2 ($\neq r_1$), and in no way depended on conditions at the rigid flow boundaries. The argument appears to be unaffected in the present case where these boundaries are not both rigid, so we would expect a necessary condition for stability still to be (1.1),

$$\Phi(r) \geqslant 0 \quad \text{everywhere.}$$

Now, even when this criterion *is* satisfied, and the interior of the fluid is essentially stable, the surface may be unstable, owing to the action of surface tension. To see this simply, let us consider a flow in which $\Phi(r)$ is everywhere positive, with an axisymmetric disturbance of wave-number k (where $k > 0$ still), and let us suppose that conditions at the surface make it only just unstable, i.e. σ^2 is negative, but very small (see § 5 for a proof that σ^2 must be real for axisymmetric disturbances). Without loss of generality, also, let the outer boundary be the free surface. If the slowly growing disturbance were to create radial displacements in the interior of the fluid, the basic rotational restoring force (cf. Rayleigh's (1916) argument) would rapidly pull them back, with a frequency proportional to the vorticity in the fluid (which is only zero when $\Phi(r) = 0$, so in our case it is non-zero) in other words much more quickly than the disturbance itself would try to grow, since $|\sigma^2|$ is very small. Thus, in this case of small negative σ^2, we should expect a negligible disturbance in the interior of the fluid, with disturbance velocities confined principally to a thin layer near the free surface, so that necessarily they will be an order of magnitude larger in the axial and azimuthal directions than in the radial direction. Hence some form of boundary-layer analysis should reveal this type of disturbance, and, as $\sigma^2 \to 0$, it should indicate a critical wave-number, below which all disturbances are unstable, and above which they are stable.

When $n = 0$, (2.2) reduces to

$$r\frac{d}{dr}\left(\frac{1}{r}\frac{d\psi}{dr}\right) - k^2\psi\left[1 - \frac{\Phi(r)}{\sigma^2}\right] = 0, \tag{3.1}$$

where $\psi = ur$. Now if we write $\sigma^2 = -\tau^2$, where $0 < \tau^2 \ll \Phi(b)$, we see that $d^2\psi/dr^2$ must be much larger than $k^2\psi$ for (3.1) to hold. Let us therefore seek a boundary-layer type of solution and replace r by $b(1-y)$. Equation (3.1) can now be simplified even further, in the region $y \ll 1$, to

$$\frac{d^2\psi}{dy^2} = \frac{4\Omega^2k^2b^2}{\tau^2}\psi, \tag{3.2}$$

where $\Phi(b)$ has been put equal to $4\Omega^2$. The boundary condition at $y = 0$, from (2.11) and (2.5), may now be written

$$\left.\frac{d\psi/dy}{\psi}\right|_{y=0} = -\frac{bk^2}{\tau^2}A_+. \tag{3.3}$$

That solution of (3.2) which does not grow exponentially as y increases is

$$\psi = C\,e^{-(2\Omega kb/\tau)y}, \tag{3.4}$$

where τ is positive and C is an arbitrary constant; and, if we put this into (3.3), we obtain the relation

$$kA_+ = 2\Omega\tau. \tag{3.5}$$

In other words, this boundary-layer solution is consistent with the physical conditions of the problem, so long as the wave-number k is related to the positive growth rate τ by equation (3.5) (remember that k also appears in A_+; see equation (1.3)). And it is consistent for two wave-numbers, for, as $\tau \to 0$, either $k \to 0$—the limit of infinite wavelength corresponding to no disturbance—or $A_+ \to 0$, which corresponds to a non-zero wave-number given by

$$1 - k^2b^2 = -\frac{\rho b V^2(b)}{\gamma}.$$

Now a wave-number k causing A_+ to be negative cannot correspond in this way to an unstable solution, whereas any wave-number causing A_+ to be positive can. Hence this simple boundary-layer analysis makes it plausible that a necessary and sufficient condition for stability to axisymmetric disturbances is that A_+ should be negative. A similar analysis holds when the inner boundary is free, leading to the condition that A_- be negative.

This limit as σ^2 tends to zero from below is of interest, as it gives us the form of that disturbance which divides stability from instability. The thickness of the boundary layer tends to zero, so that there is no disturbance in the interior of the fluid, i.e. all disturbance quantities tend to zero, except for velocities actually in the surface itself, and the condition $A_{\pm} \to 0$ merely indicates that there tends to be a balance between the centrifugal force on a fluid particle on the disturbed free surface (e.g. the point X of figure 1), and the restoring force exerted on it by surface tension.

It is interesting to note that the equations and boundary conditions also seem to admit of an axisymmetric solution for which all disturbance velocities are zero and for which σ is zero (as long as $A_{\bullet}$ is also zero) but for which δ is non-zero. This would be a completely steady disturbance, with the original velocity distribution everywhere. But (for a free outer boundary) a ring of fluid at the point X (figure 1) has a greater circulation than any ring of fluid in the undisturbed state, which contradicts Kelvin's circulation theorem for inviscid fluids. In other words, this disturbance is not one which can arise from the basic state, and hence cannot exist. For a viscous fluid, however, the boundary-layer situation, with violent shear in a thin region, would be physically improbable, and it is this static disturbance which is likely to be the critical one.

It is perhaps also worth remarking that the boundary-layer analysis presented above shows us the reason why the criterion for stability (to axisymmetric disturbances at least), given that the basic flow is stable, involves only the velocity at the free surface, and no other parameter of the velocity distribution. It is because, for a nearly neutral disturbance, only the layers of fluid very near the surface are perturbed, so that the basic velocity gradient, for instance, is irrelevant.

The above discussion has indicated the probable form of the stability criteria, at least for axisymmetric disturbances. The following sections will verify these results, by an exact analysis, and will point the way to some conclusions which it may be possible to derive for non-axisymmetric disturbances.

4. A sufficient condition for stability

In this section we shall follow the methods of Howard & Gupta (1962, § 3) for a general disturbance, noting that care must be taken when integrating by parts because of the boundary condition (2.11). First of all, assume that an unstable mode exists, i.e. that α and hence σ has a negative imaginary part, so that σ cannot be zero. Let us now change the dependent variable in equation (2.2), writing $u = \sigma^{\frac{1}{2}}F/r$ (choosing a particular branch of $\sigma^{\frac{1}{2}}$) the equation becomes

$$\frac{d}{dr}\left(\frac{S}{r}\sigma\frac{dF}{dr}\right)-\frac{F}{r}\left\{\sigma+2nr\frac{d}{dr}\left(\frac{SV}{r^3}\right)+\tfrac{1}{2}r\frac{d}{dr}\left(\frac{S}{r}\frac{d\sigma}{dr}\right)+\frac{1}{\sigma}S\left[\frac{1}{4}\left(\frac{d\sigma}{dr}\right)^2-k^2\Phi\right]\right\}=0. \tag{4.1}$$

If we multiply this by F_*, the complex conjugate of F, and integrate from a to b (where a may be zero, finite, or infinite), we obtain

$$\left[\frac{S}{r}\sigma\frac{dF}{dr}F_*\right]_a^b = \int_a^b \sigma\left(S\left|\frac{dF}{dr}\right|^2+|F|^2\right)\frac{dr}{r}$$
$$+\int_a^b\left[2n\frac{d}{dr}\left(\frac{SV}{r^3}\right)+\frac{1}{2}\frac{d}{dr}\left(\frac{S}{r}\frac{d\sigma}{dr}\right)\right]|F|^2\frac{dr}{r}$$
$$+\int_a^b\frac{1}{\sigma}S\left[\frac{1}{4}\left(\frac{d\sigma}{dr}\right)^2-k^2\Phi\right]|F|^2\frac{dr}{r}. \tag{4.2}$$

In order to evaluate the left-hand side of this equation, we must write the boundary conditions (2.6) and (2.11) in terms of F. (2.6) becomes

$$\left.\begin{aligned} F &= o(r), \quad r = 0,\\ F &= 0, \quad r = a \text{ (finite)},\\ F &= o(r^{\frac{1}{2}}) \quad r\to\infty, \end{aligned}\right\} \tag{4.3}$$

so that the quantity in the square bracket on the left of (4.2) is zero at $r = a$ for all a. The condition (2.11) becomes (using (2.5))

$$\left.\frac{dF/dr}{F}\right|_{r=b} = \mp\frac{A_\pm}{\sigma^2 S}+\frac{1}{\sigma}\left(\frac{1}{2}\frac{d\sigma}{dr}+\frac{2nV}{r^2}\right) \tag{4.4}$$

the right-hand side also being evaluated at $r = b$. Thus the left-hand side of (4.2) is

$$\frac{1}{r}|F|^2\left\{\mp\frac{A_\pm}{\sigma}+S\left(\frac{1}{2}\frac{d\sigma}{dr}+\frac{2nV}{r^2}\right)\right\} \tag{4.5}$$

evaluated at $r = b$.

Now, we have assumed that σ has a non-zero imaginary part. Noticing that $d\sigma/dr$ is real, let us take the imaginary part of equation (4.2), to obtain

$$\left[\pm\frac{A_\pm}{b}\left|\frac{F}{\sigma}\right|^2\right]_{r=b} = \int_a^b\left(S\left|\frac{dF}{dr}\right|^2+|F|^2\right)\frac{dr}{r}+\int_a^b S\left[k^2\Phi-\frac{1}{4}\left(\frac{d\sigma}{dr}\right)^2\right]\left|\frac{F}{\sigma}\right|^2\frac{dr}{r}. \tag{4.6}$$

When $a < b$ (free outer boundary) both terms on the right-hand side are positive, as long as

$$k^2\Phi \geqslant \frac{1}{4}\left(\frac{d\sigma}{dr}\right)^2 \tag{4.7}$$

for all r. The upper sign must be taken on the left-hand side, which is therefore negative if A_+ is negative. Thus a sufficient condition for (4.6) to have only the trivial solution $F \equiv 0$, i.e. a sufficient condition for stability, is that both (4.7) is satisfied, and $A_+ \leqslant 0$. Similarly, when $a > b$, a sufficient condition for stability is that both (4.7) is satisfied, and $A_- \leqslant 0$.

If $d\sigma/dr = n\,d(V/r)/dr$ should happen to be identically zero, the condition (4.7) reduces to Rayleigh's criterion, $\Phi(r) \geqslant 0$ everywhere. Thus the joint condition

$$\Phi(r) \geqslant 0 \quad \text{everywhere, and} \quad A_\pm \leqslant 0$$

is a sufficient condition for stability if *either* $n = 0$, i.e. the disturbance is axisymmetric, *or* $d(V/r)/dr \equiv 0$, i.e. $V = \Omega r$ (solid-body rotation), and then n may be non-zero.

For a non-axisymmetric disturbance to a *general* basic flow, however, the condition (4.7) does not reduce to the simple one we had hoped for. In general, (4.7) may be written

$$0 \leqslant \frac{d}{dr}\left(\frac{V}{r}\right) \leqslant \frac{8k^2r^3V(r)}{n^2}, \tag{4.8}$$

which, first, requires that the angular velocity of the basic flow either is constant or increases outwards, and, secondly, if the angular velocity does increase, (4.8) puts bounds on the values of k/n which are possible for unstable disturbances. However, since in many flows of interest the angular velocity decreases outwards (for example, any flow with a hollow core which tends to the form of a potential vortex at large radius), and, since the fact that a sufficient condition for stability is not satisfied does not mean that the flow is unstable, we can see that (4.8) yields little useful information. The author, in common with those interested in the stability of Couette flow between rigid cylinders, has been unable to find a general necessary and sufficient condition for stability to non-axisymmetric disturbances. The only way in which any advance appears to be possible is by means of a separate numerical solution to the eigenvalue problem of (2.2), (2.6) and (2.11) for every case of interest.

The analysis of this section may be extended to investigate the stability of the interface between two fluids of different but comparable densities, when the undisturbed tangential velocity is continuous across the interface (which is always required in practice). Alterman (1961) looked at this problem when each fluid was in solid-body rotation (and the angular velocities were different), but restricted her attention to axisymmetric disturbances. She obtained a necessary, and (although she did not realize it) sufficient condition for stability. In the present case, we may use the above methods (with more complicated algebra) to show that sufficient conditions for stability are (i) that (4.7) should be satisfied by the undisturbed flow in *each* fluid, and (ii) that the quantity

$$A = \frac{\gamma}{b^2}\left\{1 - n^2 - k^2b^2 + \frac{(\rho_1 - \rho_2)\,b\,V^2(b)}{\gamma}\right\}$$

should be negative or zero, where ρ_1 is the density of the inner fluid, and ρ_2 that of the outer fluid. This reduces to

$$\Phi \geqslant 0 \quad \text{in each fluid, and} \quad A \leqslant 0$$

both when $n = 0$ (axisymmetric disturbances) and when $V = \Omega r$ (solid-body rotation in each fluid). The details of the calculation are given in Pedley (1966, § 3).

5. A necessary condition for stability to axisymmetric disturbances

In this section it is demonstrated rigorously that the sufficient condition, found in § 4, for stability to axisymmetric disturbances ($n = 0$) is also a necessary condition, confirming the plausible arguments of § 3. When $n = 0$, σ is constant ($= \alpha$), S is constant ($= 1/k^2$), and considerable simplification results. For one thing, any eigenvalue σ^2 must be real, because now the only difference between (4.2) and its complex conjugate (for real k) is that σ^2 is replaced by its complex conjugate, with the result that either F is identically zero, or σ^2 is real. Hence, for a real disturbance, σ^2 is real.

Now we must look in more detail at the solutions of (2.2), which, setting $ru = \psi$, may be written

$$r\frac{d}{dr}\left(\frac{1}{r}\frac{d\psi}{dr}\right) - k^2\psi\left[1 - \frac{\Phi(r)}{\sigma^2}\right] = 0. \tag{5.1}$$

It is convenient to change the independent variable to $x = r^2/b^2$, so that (5.1) becomes

$$\frac{d^2\psi}{dx^2} = \frac{k^2b^2}{4}\frac{\psi}{x}\left[1 - \frac{\phi(x)}{\sigma^2}\right], \tag{5.2}$$

where $\phi(x) \equiv \Phi(r)$. The boundary conditions also change: (2.6) becomes

$$\left.\begin{aligned} &\psi = o(x^{\frac{1}{2}}) \quad \text{as} \quad x \to 0, \\ &\psi = 0 \quad \text{when} \quad x = x_0 = (a/b)^2, \\ &\psi = o(x^{\frac{1}{2}}) \quad \text{as} \quad x \to \infty. \end{aligned}\right\} \tag{5.3}$$

From (2.5), we have

$$i\sigma p = \frac{\sigma^2}{rk^2}\frac{d\psi}{dr} = \frac{2\sigma^2}{k^2b^2}\frac{d\psi}{dx},$$

so that (2.11) becomes

$$\left.\frac{d\psi/dx}{\psi}\right|_{x=1} = \mp\frac{k^2b}{2\sigma^2}A_{\pm}. \tag{5.4}$$

We may note that, although equation (5.2) is typical of Sturm–Liouville theory, the problem is *not* a Sturm–Liouville problem, because the eigenvalue σ^2 occurs both in the equation (5.2) *and* in the boundary condition (5.4). For this reason we cannot, as in the case of rigid boundaries, apply a standard piece of theory and quote the result; we must develop a separate theory for our particular circumstances.

From now on it will be necessary to classify flows according to the sign of $\Phi(r)$. The easiest case is that of potential vortex flow, for which $\Phi(r) \equiv 0$. This case was solved completely by Ponstein (1959), and his results are given in table 1. All other cases are presented below in this section.

(i) $\Phi(r) \geqslant 0$

In this case $\phi(x) \geqslant 0$ everywhere, and we require that there is a finite range of x, a subset of the range $[1, x_0]$ (where x_0 may be zero, finite, or infinite), over which $\phi(x)$ is non-zero and positive. From (5.2), we immediately see that, if σ^2 is negative, then $d^2\psi/dx^2$ has everywhere the same sign as ψ. Thus, when the outer boundary is free, and $x_0 < 1$ (exactly similar arguments hold for $x_0 > 1$) and if ψ is taken (without loss of generality) to be positive in the neighbourhood of x_0, the graph of ψ is everywhere concave upwards (see figure 2, full curve). Hence the left-hand side of (5.4) is always positive, and for consistency A_+ must be positive. This not only confirms what we already knew from § 4—that $A_+ > 0$ is a necessary condition for σ^2 to be negative (i.e. for instability)—but it also demonstrates that, for *every* negative σ^2, there exists a single, positive A_+ (or A_- when $x_0 > 1$) for which the problem has a solution. Thus A_+ can be regarded as a positive, single-valued function of σ^2, if $-\infty < \sigma^2 < 0$.

In order for what follows to be valid, a certain further restriction must now be imposed on $\phi(x)$: we require that it be bounded, or, mathematically,

$$0 \leqslant \phi(x) \leqslant 4\Omega^2 \quad \text{(say)} \tag{5.5}$$

for all x in the range $[1, x_0]$. Since

$$\phi(x) \equiv \Phi(r) = \frac{1}{r^3}\frac{d}{dr}(r^2V^2) = \frac{2V\omega}{r},$$

where $\omega(r)$ is the axial vorticity; and, since both as $r \to 0$ and as $r \to \infty$ the condition that ω should be bounded implies that $V = O(r)$, the constraint (5.5) is equivalent to the condition of bounded vorticity in the undisturbed flow for all possible values of x_0. Such a condition is always satisfied in a real flow.

With this restriction on $\phi(x)$ we may now proceed. We wish to prove that $A_\pm \leqslant 0$ is a necessary condition for stability, or in other words that a sufficient condition for instability is $A_\pm > 0$. Let us once more consider only the case when the outer boundary is free, and then the problem may be restated as follows: given an arbitrary positive value of A_+, is there a solution of equation (5.2), satisfying boundary conditions (5.3) and (5.4), for *some* negative value of σ^2? Or again: we know that A_+ can be regarded as a positive function of σ^2 (for $\sigma^2 < 0$); does it necessarily take *all* positive values as σ^2 varies between $-\infty$ and zero?

We shall answer the question as posed in its latter form. The condition on the inner boundary $x = x_0$ is that $\psi = 0$. [Note that, when the inner boundary is free, and $x_0 = \infty$, (5.3) appears to allow a non-zero $\psi = o(x^{\frac{1}{2}})$ as $x \to x_0$; in fact all solutions of (5.2), for bounded $\phi(x)$, take the form

$$\psi \sim x^{\frac{1}{2}} K_1(k' x^{\frac{1}{2}})$$

as $x \to \infty$, where k' is proportional to k, and K_1 is a modified Bessel function of the second kind, so that ψ always tends to zero exponentially.] The outer boundary condition (5.4) may also be slightly rewritten. It tells us nothing about the *value* of ψ at $x = 1$, only about the ratio of $d\psi/dx$ to that value, so that by appropriate scaling we can always demand that $\psi(1) = 1$, and then A_+ is related to the slope by

$$A_+ = \frac{-2}{k^2 b}\psi'(1)\,\sigma^2$$

(where the prime denotes differentiation with respect to x). Thus what we have to do is to solve (5.2) subject to

$$\psi(x_0) = 0, \quad \psi(1) = 1, \tag{5.6}$$

for all negative values of σ^2. Then $\psi'(1)$ will be a function of σ^2, with a positive value (see figure 2). Let us write

$$\frac{2}{k^2 b}\psi'(1) = G(\sigma^2).$$

Now we shall prove that

$$A_+ \equiv -\sigma^2 G(\sigma^2)$$

takes all positive values as σ^2 varies from $-\infty$ to 0; and the proof may be performed in three stages.

(1) $G(\sigma^2)$, and hence A_+, is a continuous function of σ^2 for $\sigma^2 \neq 0$. The proof is standard and will be omitted.

(2) As $\sigma^2 \to -\infty$, $A_+ \to +\infty$. When $(-\sigma^2)$ is very large, the equation (5.2) reduces to

$$\psi'' = \frac{k^2 b^2}{4} \frac{\psi}{x}, \tag{5.7}$$

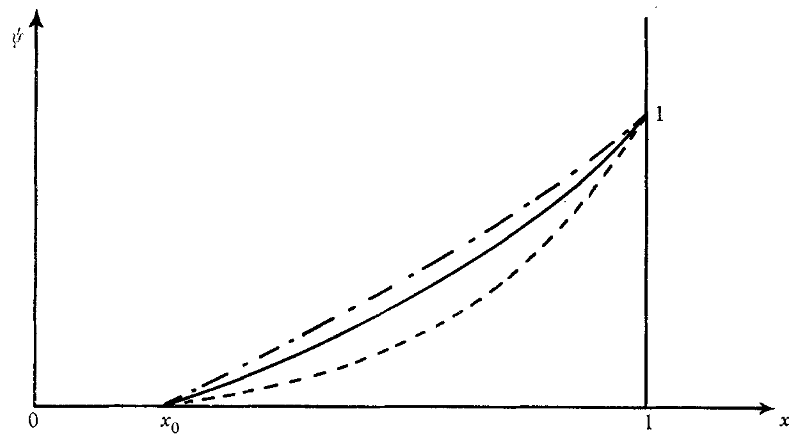

FIGURE 2. Comparison of the graphs of $\psi(x)$ for given boundary conditions, and three forms of $\phi(x)$, for the case of the free outer boundary: ————, $\phi(x)$ general, $0 \leqslant \phi(x) \leqslant 4\Omega^2$(*); —·—·—·—, $\phi(x) \equiv 0$; ---------, $\phi(x) \equiv 4\Omega^2$ defined by (*).

which is the equation for the case $\phi(x) \equiv 0$. Reverting to the variable r, we see from (5.1) that the solution of (5.7) satisfying the boundary conditions (5.6) is

$$\psi = \frac{r}{b} \frac{K_1(kr) I_1(ka) - I_1(kr) K_1(ka)}{K_1(kb) I_1(ka) - I_1(kb) K_1(ka)}$$

(this curve is the dash-dot curve in figure 2), whence

$$A_+ = \frac{\sigma^2}{k} \frac{K_0(kb) I_1(ka) + I_0(kb) K_1(ka)}{K_1(kb) I_1(ka) - I_1(kb) K_1(ka)}$$

(from (5.4)), which is positive if $a < b$, and which tends to infinity as $\sigma^2 \to -\infty$.

(3) A_+ may be made arbitrarily small, by an appropriate choice of σ^2. This follows from a comparison between the solution $\psi(x)$ of (5.2) as it stands, and the solution $\psi_\Omega(x)$ of (5.2) when $\phi(x)$ is replaced by an upper bound $4\Omega^2$ (see (5.5)). Everywhere in the range $[x_0, 1]$

$$\frac{\psi''}{\psi} \leqslant \frac{\psi''_\Omega}{\psi_\Omega},$$

so that the curve of $\psi_\Omega(x)$ lies everywhere below that of $\psi(x)$ (see figure 2, broken and full curves respectively), and in particular, the slope of $\psi_\Omega(x)$ at $x = 1$ is greater in magnitude than that of $\psi(x)$. In other words,

$$0 < G(\sigma^2) \leqslant G_\Omega(\sigma^2),$$

so that
$$0 < -\sigma^2 G(\sigma^2) \leqslant -\sigma^2 G_\Omega(\sigma^2)$$

(since $\sigma^2 < 0$). Now $-\sigma^2 G_\Omega(\sigma^2)$ is just what we might call $A_+^{(\Omega)}$, the value of A_+ for this σ^2 in the case $\phi(x) \equiv 4\Omega^2$. So, if we can prove that, when $\phi(x) \equiv 4\Omega^2$, there exist negative values of σ^2 which permit $A_+^{(\Omega)}$ to take arbitrarily small values, we shall then have shown that A_+ takes arbitrarily small values, because

$$A_+ = -\sigma^2 G(\sigma^2) \leqslant A_+^{(\Omega)}.$$

But we have essentially proved this already, by the boundary-layer solution of § 3. As $\sigma^2\,(= -\tau^2)$ tends to zero from below, the solution for ψ_Ω satisfying the boundary conditions (5.6) is approximately given by (3.4) with $C = 1$, and $A_+^{(\Omega)}$ is related to τ by (3.5), viz.
$$kA_+^{(\Omega)} = 2\Omega\tau.$$

Thus, as τ tends to zero, $A_+^{(\Omega)}$ also tends to zero, and there exist negative values of σ^2 for which $A_+^{(\Omega)}$ becomes arbitrarily small. Hence assertion (3) above is proved. [The proof that $A_+^{(\Omega)}$ takes arbitrarily small values need not be given approximately in this way, since the equation for ψ_Ω can be solved exactly, and the boundary condition (5.4), expressed as an equation for σ^2, can be shown to have a solution for every positive $A_+^{(\Omega)}$. However, the analysis is tiresome, and the above argument, although approximate, is none the less valid.]

We have proved that A_+ is a continuous positive function of σ^2, which may take values both arbitrarily large and arbitrarily small for negative σ^2, completing the required proof that $A_+ \leqslant 0$ is a necessary condition for stability to axisymmetric disturbances when the outer boundary is free, and $0 \leqslant \phi(x) \leqslant 4\Omega^2$. A completely parallel proof holds when the inner boundary is free, and the necessary condition is then $A_- \leqslant 0$.

(ii) $\Phi(r) \leqslant 0$

The first result of (i) above has an immediate parallel here; viz. if σ^2 is positive, then $A_\pm < 0$. This means that $A_\pm < 0$ is a necessary condition for stability. But it is not a sufficient condition for stability, because there may be negative values of $A_\pm$ which permit negative as well as positive values of σ^2 (in (i) above it did not matter if for positive $A_\pm$ there were possible positive values of σ^2; there was one negative value, therefore instability had to follow). We shall indeed prove that, for every *negative* $A_\pm$, there is at least one *negative* characteristic value for σ^2, as well as a positive value, so that instability is possible for all $A_\pm$, and hence $\Phi(r) \leqslant 0$ always leads to instability. Take, therefore, an arbitrary negative $A_\pm$, and see whether it can lead to a negative value for σ^2.

If we can show both that for two different negative values of σ^2 (say σ_0^2 and σ_2^2) (5.2) has a solution satisfying (5.3) *and* the condition $\psi(1) = 0$, and that, for just one intermediate value of σ^2 (say σ_1^2), there is a solution satisfying $\psi'(1) = 0$, then we can see that there exists one value of σ^2 (lying between σ_0^2 and σ_2^2) for which (5.4) holds, with arbitrary $A_\pm < 0$. For consider figure 3, which contains the graphs of $y = \psi'(1)/\psi(1)$ and $y = \pm\frac{1}{2}k^2 bA_\pm/(-\sigma^2)$ as functions of $(-\sigma^2)$, at

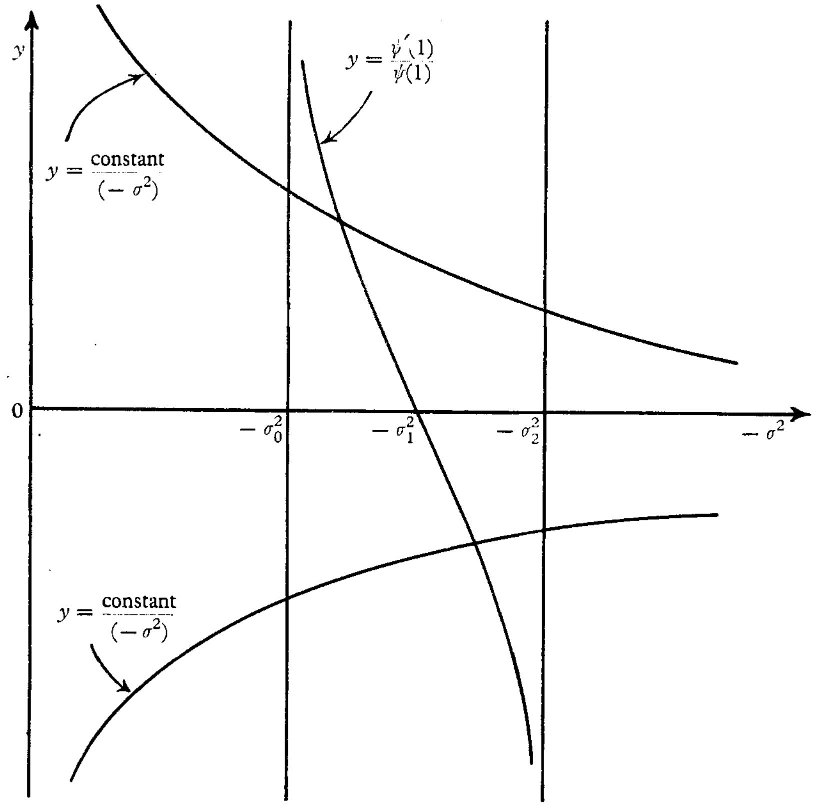

FIGURE 3. If $\psi(1)$, as a function of σ^2, has at least 2 zeros (σ_0^2 and σ_2^2), and if $\psi'(1)$ has just one intermediate zero (σ_1^2), then the equation $\psi'(1)/\psi(1) = \text{constant}/(-\sigma^2)$ always has a solution.

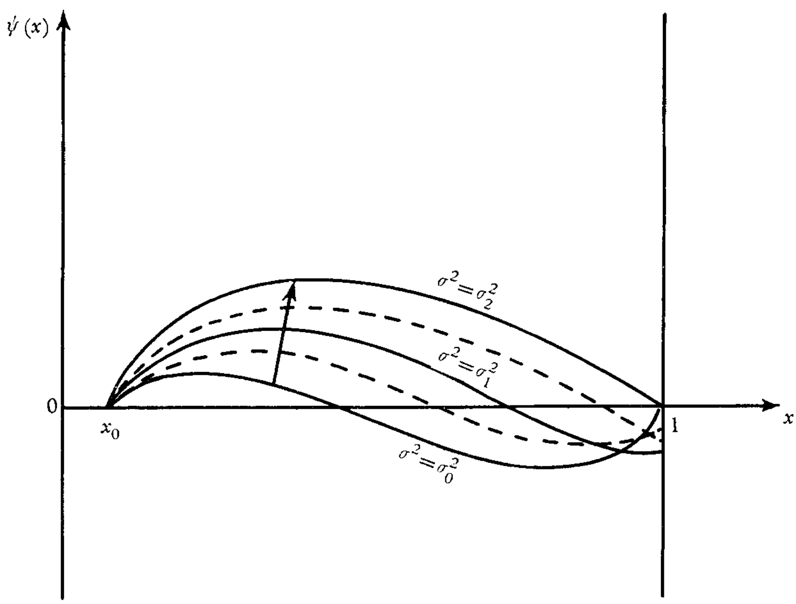

FIGURE 4. The sequence of curves $\psi(x)$ as σ^2 is increased from σ_0^2 to σ_2^2. An intermediate value of σ_1^2 is found for which $\psi'(1) = 0$.

least between $-\sigma_0^2$ and $-\sigma_2^2$. $\psi'(1)/\psi(1)$ is infinite at $\sigma^2 = \sigma_0^2$ and σ_2^2, and, since it changes sign once in between (at σ_1^2), it must take all values between $\pm\infty$ while σ^2 lies between σ_0^2 and σ_2^2. Thus the curve $y = \text{constant}/(-\sigma^2)$, whatever the sign of the constant, must cut the curve of $y = \psi'(1)/\psi(1)$ once, for σ^2 between σ_0^2 and σ_2^2.

Since the solutions of (5.2) satisfying (5.3) are continuous functions of σ^2, it will be sufficient to show that there exists a value of $\sigma^2(\sigma_0^2)$ for which $\psi(x)$ has one zero at $x = 1$, and one further zero for x between x_0 and 1. If then $(-\sigma^2)$ is increased, a value will be reached $(-\sigma_1^2)$ for which $\psi'(1) = 0$; and, if it is increased still further, a value will be reached for which $\psi(1) = 0$ again (see figure 4, drawn for the case $x_0 < 1$). The first zero of $\psi(x)$ away from x_0 moves continuously from a position between 1 and x_0, in the direction of 1, until in the end it reaches 1. That such negative values of σ^2 do exist when $\phi(x) \leqslant 0$ follows from standard Sturm–Liouville theory, for now we have a real Sturm–Liouville problem, with $\psi(x) = 0$ at each end of the considered range, and no σ^2 in the boundary conditions (see, e.g., Burkill 1956, chapter III). The proof is based on the fact that a sufficient condition for there to be at least N zeros of $\psi(x)$ in a range (x_1, x_2) is

$$-\max\left\{\frac{k^2b^2}{4x}\left(1-\frac{\phi(x)}{\sigma^2}\right)\right\} \geqslant \frac{N^2\pi^2}{(x_2-x_1)^2} \tag{5.8}$$

(this follows from the corollary to Burkill's theorem 13) so, if there is a finite range (x_1, x_2) within our range $[x_0, 1]$ in which $\phi(x) \leqslant -\delta < 0$, where δ is arbitrarily small, then $(-\sigma^2)$ may always be chosen sufficiently small for (5.8) to be satisfied with any N.

Thus we have proved that to every $A_\pm$ there corresponds at least one unstable solution, and therefore, when $\Phi(r) \leqslant 0$, all disturbances are unstable. This confirms the expectation voiced at the start of this section.

(iii) *The case where $\Phi(r)$ changes sign*

In this case also, as in the case with rigid boundaries, all flows are unstable, as long as there is a finite region within which $\Phi(r)$ is less than zero. The arguments surrounding equation (5.8) show that there exist negative values of σ^2 which will cause $\psi(x)$ to have at least two zeros within this region; whence, as in the last subsection, the graphs of $y = \psi'(1)/\psi(1)$ and $y = \pm\frac{1}{2}k^2bA_\pm/(-\sigma^2)$ as functions of $(-\sigma^2)$ must intersect at least once. We have thus completed the proof that a necessary and sufficient condition for a general rotating flow (excluding Ponstein's case $\Phi(r) \equiv 0$) to be stable to axisymmetric disturbances is that both $\Phi(r) \geqslant 0$ for all r in $[a, b]$, and $A_\pm \leqslant 0$.

6. The 'most unstable disturbance' in the case of solid body rotation: $\Phi(r) \equiv 4\Omega^2$

The analysis of § 4 has shown that the case of solid-body rotation is a somewhat special one, in that the condition $A_\pm \leqslant 0$ is sufficient for stability even to non-axisymmetric disturbances. Now, when the inner boundary is free, this condition requires that

$$\rho b^3\Omega^2/\gamma \geqslant 1-n^2-k^2b^2, \tag{6.1}$$

demonstrating that non-axisymmetric disturbances must be stable. But § 5 shows us that (6.1), with $n = 0$, is also a necessary condition for stability; so that here, as in the case of potential vortex flow, $A_- \leqslant 0$ is indeed both a necessary and a sufficient condition for stability to all disturbances.

When the outer boundary is free, on the other hand, $A_+ \leqslant 0$ requires that

$$-\rho b^3\Omega^2/\gamma \geqslant 1-n^2-k^2b^2 \tag{6.2}$$

so that non-axisymmetric disturbances may be unstable. [Moreover, they are not covered by the analysis of § 5, and so the flow may be stable even when (6.2) is not satisfied; indeed Hocking & Michael (1959) showed that plane disturbances ($k = 0$) *are* stable unless

$$\rho b^3\Omega^2/\gamma \leqslant n(n+1),$$

see table 1.] It would therefore be of interest to work out whether instability might ever occur for a non-axisymmetric disturbance: i.e. if such a disturbance is ever the most unstable one, having a larger growth rate than any other. Ponstein (1959) showed that this indeed occurs in the potential vortex case, and some experiments by L. de Jong (private communication) indicate that it probably does in the solid-body case too.

In order to calculate the most unstable disturbance, it is necessary to know the exact form of a general unstable disturbance, which means solving the equations exactly. In the case of solid-body rotation, this can be done, for, if u, instead of p, is eliminated from the equations, the resulting single equation for $p(r)$ is a modified Bessel's equation of order n (see Phillips 1960, equation (4.6)). Unfortunately, calculation of the growth rate of a general unstable disturbance is still not possible in simple analytical terms, and it is only in the two extreme cases of plane disturbances ($k = 0$) and axisymmetric disturbances ($n = 0$) that growth rates can be computed. However, this is enough for our purposes, for, if we can find a range of values of the physical parameters of the problem, for which the most unstable plane disturbance has a higher growth rate than the most unstable axisymmetric disturbance, then we shall have demonstrated that non-axisymmetric disturbances can indeed be the ones to appear in certain unstable situations.

For convenience, and without loss of generality, we take the inner boundary of the flow to be $r = 0$. When $k = 0$, that solution of the equation for $p(r)$ which is regular at $r = 0$ is $p = \text{constant} \times r^{|n|}$ and the boundary condition (2.11) on $r = b$ becomes (after A_+ is written out in full, and dropping the modulus signs round n)

$$(\sigma-\Omega)^2 = \frac{\gamma n(n^2-1)}{\rho b^3} - (n-1)\,\Omega^2$$

(cf. Hocking & Michael 1959). This shows that σ can have a non-zero imaginary part (for positive integer n), only if both $n \geqslant 2$ and

$$1-Ln(n+1) > 0,$$

where

$$L = \gamma/\rho b^3\Omega^2, \tag{6.3}$$

and, in that case, the imaginary part of σ (written $\beta\Omega$ for convenience) is given by

$$\beta^2 = \beta_1^2 \equiv (n-1)\,[1-Ln(n+1)]. \tag{6.4}$$

Bearing in mind that $n \geqslant 2$, this shows that, if $L < \frac{1}{6}$, an unstable plane disturbance is possible.

The case of $n = 0$ is a little more difficult, although we know σ^2 to be real. That solution of equation (5.1) which is regular at $r = 0$ is

$$\psi = C\frac{rI_1(k'r)}{bI_1(k'b)},$$

where

$$k'^2 = k^2\left(1-\frac{4\Omega^2}{\sigma^2}\right) \tag{6.5}$$

and C is a constant. This time the boundary condition (2.11) becomes

$$\frac{k'bI_1(k'b)}{I_0(k'b)} = \frac{4-\sigma^2/\Omega^2}{1+L(1-k^2b^2)}. \tag{6.6}$$

Writing $\sigma^2 = -\beta^2\Omega^2$ for an unstable disturbance, and replacing $k'b/k$ by z, (6.5) becomes

$$\beta^2 = \frac{4b^2}{z^2-b^2} \tag{6.7}$$

and (6.6) becomes

$$k[1+L(1-k^2b^2)]\frac{I_1(kz)}{I_0(kz)} = \frac{4z}{z^2-b^2}. \tag{6.8}$$

(6.7) tells us that maximizing $|\beta|$ is equivalent to minimizing z, so we must (in principle) solve (6.8) for z, and minimize it as k is varied. If, at the minimum of z, kz is sufficiently large, the quantity $I_1(kz)/I_0(kz)$ will be approximately unity (indeed, this is valid to within 10 % if kz is greater than about 6); we shall therefore assume it to be so, and verify it afterwards. (6.8) thus becomes

$$k[1+L(1-k^2b^2)] = \frac{4z}{z^2-b^2}. \tag{6.9}$$

To find the minimum of z, differentiate (6.9) with respect to k, put dz/dk equal to zero, and obtain

$$k = \frac{1}{b}\left(\frac{1+L}{3L}\right)^{\frac{1}{2}}. \tag{6.10}$$

At this value of k, (6.9) gives z (which must be greater than b) to be

$$z = b\,\frac{(27L)^{\frac{1}{2}}+[27L+(1+L)^3]^{\frac{1}{2}}}{(1+L)^{\frac{3}{2}}}, \tag{6.11}$$

which is evidently a minimum since, as $kb \to [(1+L)/L]^{\frac{1}{2}}$ in (6.9), $z \to \infty$. The above condition that the value of kz corresponding to this solution should be greater than about 6 implies that

$$1+3L-105L^2-107L^3 > 0,$$

which is certainly satisfied if $L \leqslant \frac{1}{10}$. Now the growth rate calculated from (6.7) and (6.11) is given by

$$\beta^2 = \beta_2^2 \equiv \frac{2(1+L)^3}{27L+\{27L(1+30L+3L^2+L^3)\}^{\frac{1}{2}}}. \tag{6.12}$$

The question we ask is whether the growth rate (6.4) for plane disturbances can be greater than the growth rate (6.12) for axisymmetric ones. It can, as long as

$$\beta_1^2 > \beta_2^2. \tag{6.13}$$

The maximum permitted value of L is $\frac{1}{10}$, so try $L = \frac{1}{20}$: the right-hand side of (6.13) is then approximately 0·725, and $n = 3$ gives the left-hand side a value 0·8. So, for this and any smaller value of L, we can find an n which will satisfy (6.13). Thus we have the final result that, if the quantity $\rho b^3\Omega^2/\gamma$ ($= 1/L$ from (6.3)) is greater than a number slightly less than 20, there is a plane disturbance more unstable than any axisymmetric disturbance, and hence instability will not occur axisymmetrically.

7. Possible applications. Effect of viscosity

Situations where the stability of swirling flows with a cylindrical free surface is of interest are frequently encountered, particularly in the field of chemical engineering. Swirl atomization of liquid jets, for instance, is a process where rotation has a destabilizing influence (free outer boundary). In general the basic swirl velocity distribution will not be solid-body rotation, owing to conditions at the nozzle where the jet is formed (far downstream from the nozzle, solid-body rotation *would* be set up by the action of viscosity, but the jet would have broken up by then anyway), and hence to consider arbitrary velocity distributions $V(r)$ is not a purely academic exercise. Note that a slow axial variation of the basic flow will not affect the stability criteria, as long as the length scale of that variation is large compared with the radial length scale b. Swirling flow in a pipe, with an air core, is a swirling flow with a free inner boundary, although in real situations there will generally be a radially varying axial velocity $W(r)$, and possibly a tangential stress on the free surface (due to the flow of air), in addition to the basic swirl. Another example where the analysis for this case might be relevant is in the cavitating line vortex behind a hydrofoil.

However, before such relatively complicated, industrial applications of the theory are made, in situations where other unforeseen phenomena may have an effect on the stability criteria, it would be worth trying to test it by means of a simple, controlled, laboratory experiment. One such experiment readily springs to mind, which should test the conclusion that, when the inner boundary of a swirling flow is free, the flow will be stable to all disturbances as long as

$$\rho b V^2(b)/\gamma > 1,$$

at least in the cases of potential vortex flow and solid-body rotation. Partially fill a closed cylindrical container with a liquid and rotate it so rapidly about a vertical axis that an air core is formed, with, after a short time, solid-body rotation in the liquid surrounding it. The shape of the air core is determined by the balance of centrifugal, gravitational and surface tension forces, and will be narrower at the bottom than at the top. Thus, if it is possible to vary either the rotation speed or the volume of liquid in the cylinder, a situation can be reached where

$$\rho b^3\Omega^2/\gamma = 1 \tag{7.1}$$

at a certain level in the cylinder. Above this level we should expect the interface to be stable, and below it we should expect it to be unstable to disturbances of long wavelength.

The radius r of the air core is given by

$$\tfrac{1}{2}\Omega^2(r^2-b^2)-\frac{\gamma}{\rho}\left(\frac{1}{r}-\frac{1}{b}\right) = -gz, \tag{7.2}$$

where the origin of z (the downwards vertical) is taken where $r = b$, and b is given by (7.1). For the stability analysis to be valid, the core must be almost cylindrical, i.e. in the neighbourhood of $r = b$

$$\left|\frac{dz}{dr}\right| \gg 1,$$

i.e.

$$\Omega^2 b+\frac{\gamma}{\rho b^2} \gg g. \tag{7.3}$$

Eliminating b from (7.3) and (7.1), we see that the angular velocity must be very large for the analysis to be valid:

$$\Omega \gg (g^3\rho/8\gamma)^{\frac{1}{4}}. \tag{7.4}$$

If the liquid is water ($\gamma \doteqdot 74\,\text{dynes}\,\text{cm}^{-1}$), the right-hand side of (7.4) works out at about 300 rev/min, which means that the rotation rate must be high, but not impossibly so.

It only remains now to discuss the effect of viscosity on the stability criteria listed in table 1. There are two possible ways in which it might be important. First, the basic swirl will in general be determined by viscosity, and is therefore likely to be essentially unsteady (unless it is a solid-body rotation). Thus, for the theory to be valid, the time-scales of the relevant disturbances must be small compared with the time-scale of changes in the basic flow—we must assume a quasi-steady situation in which the Reynolds number of the basic flow is large. Secondly, and more important, viscosity will affect the disturbances themselves. In many circumstances one expects viscosity to be a stabilizing influence—for instance, the criterion for stability of a cylindrical free surface without rotation is unaffected by viscosity, which merely damps out stable oscillations, and decreases the growth rate of unstable ones (Rayleigh 1892). In the case of rotating flows between rigid cylinders, the stability criterion for axisymmetric disturbances is altered, in that certain flows, unstable by the inviscid analysis, are stabilized (Taylor 1923). The cases where a small viscosity has a *de*stabilizing influence are usually those where the phase velocity of the disturbance has a component parallel to the basic flow (e.g. plane parallel shear flows in the presence of a wall). Thus in our problem we should expect viscosity to have a destabilizing influence only on non-axisymmetric disturbances, for which the phase velocity has an azimuthal component: this is indeed found to be the case when the basic swirl is a solid-body rotation. In a series of papers by Hocking (1960), Gillis (1961) Gillis & Kaufman (1961), Gillis & Suh (1962), the viscous stability problem, when $V = \Omega r$, has been completely solved by means of a combined analytical and numerical treatment, at least for the case of a free outer boundary (a similar treatment no doubt works for the other case, but all non-axisymmetric disturbances are essentially stable then anyway) and it is shown that, for any disturbance, and any viscosity, a necessary and sufficient condition for stability is $A_\pm \leqslant 0$. Thus the criterion for stability to axisymmetric disturbances is unaltered, but certain plane disturbances (and presumably certain general non-axisymmetric disturbances also, although no inviscid criterion is known for them) are destabilized. The viscous result, that the necessary and sufficient stability criterion reduces to $A_\pm \leqslant 0$ for *all* disturbances, is refreshingly simple, but we must remember that it has been obtained only for solid-body rotation, the sole basic flow itself unaffected by viscosity, and the analysis cannot be extended to other situations. However, the result does have some relevance to other basic flows, since viscosity will always cause them to tend (slowly) to a solid-body rotation in the neighbourhood of the free surface $r = b$, and, as long as $\Phi(r)$ is everywhere positive, the stability criterion depends only on the velocity near $r = b$. So we might expect the stability of real fluids in a real experimental set-up to depend only on the sign of $A_\pm$ (and the sign of $\Phi(r)$, of course). This surmise, however, has no rigorous justification.

The author would like to record his thanks to Professor G. K. Batchelor of the Department of Applied Mathematics and Theoretical Physics, Cambridge University, for his help and encouragement throughout the course of this research, which was performed while the author was in receipt of a Research Studentship from the Science Research Council of Great Britain. Final preparation and revision of the manuscript took place with the support of the U.S. Office of Naval Research, contract number Nonr 4010(02).

REFERENCES

ALTERMAN, Z. 1961 *Phys. Fluids*, **4**, 955.

BURKILL, J. C. 1956 *The Theory of Ordinary Differential Equations*. Edinburgh: Oliver and Boyd.

GILLIS, J. 1961 *Proc. Camb. Phil. Soc.* **57**, 152.

GILLIS, J. & KAUFMAN, B. 1961 *Quart. Appl. Math.* **19**, 301.

GILLIS, J. & SUH, K. S. 1962 *Phys. Fluids*, **5**, 1149.

HOCKING, L. M. 1960 *Mathematika*, **7**, 1.

HOCKING, L. M. & MICHAEL, D. H. 1959 *Mathematika*, **6**, 25.

HOWARD, L. N. & GUPTA, A. S. 1962 *J. Fluid Mech.* **14**, 463.

LIN, C. C. 1955 *The Theory of Hydrodynamic Stability*. Cambridge University Press.

PEDLEY, T. J. 1966 Ph.D. Dissertation, Cambridge University, Chapter VI.

PHILLIPS, O. M. 1960 *J. Fluid Mech.* **7**, 340.

PONSTEIN, J. 1959 *Appl. Sci. Res.* A **8**, 425.

RAYLEIGH, LORD 1879 *Proc. Lond. Math. Soc.* **10**, 4.

RAYLEIGH, LORD 1892 *Phil. Mag.* **34**, 145.

RAYLEIGH, LORD 1916 *Proc. Roy. Soc.* A **113**, 148.

ROSENTHAL, D. K. 1962 *J. Fluid Mech.* **12**, 358.

TAYLOR, G. I. 1923 *Phil. Trans.* A **223**, 289.